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$$\begin{aligned} [\hat{x}, \frac{\hat{d}}{dx}] \psi &= (x \frac{d}{dx} \psi - \frac{d}{dx} (x\psi)) = x \frac{d\psi}{dx} - \psi - x \frac{d\psi}{dx} = -\psi \\ [x, \frac{d}{dx}] &= -1 \end{aligned} \quad (1.1)$$

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$$\begin{aligned} [M_x, y] \psi &= (M_x(y\psi) - yM_x\psi) = ((y\hat{p}_z - z\hat{p}_y)(y\psi) - y(y\hat{p}_z - z\hat{p}_y)\psi) = \\ &= -i\hbar \left(\left(y^2 \frac{\partial \psi}{\partial z} - z \frac{\partial}{\partial y} (y\psi) \right) - y \left(y \frac{\partial \psi}{\partial z} - z \frac{\partial \psi}{\partial y} \right) \right) = \\ &= -i\hbar \left(\left(y^2 \frac{\partial \psi}{\partial z} - z\psi - zy \frac{\partial \psi}{\partial y} \right) - y^2 \frac{\partial \psi}{\partial z} - yz \frac{\partial \psi}{\partial y} \right) = \\ &= -i\hbar(-z\psi) = iz\psi \\ [\hat{M}_x, \hat{y}] &= i\hat{z} = i\varepsilon_{xyz}\hat{z} \end{aligned} \quad (1.2)$$

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$$[\hat{M}^2, \hat{r}] \quad (1.3)$$

$$[\hat{M}^2, \hat{x}] \psi = ((\hat{M}_x^2 + \hat{M}_y^2 + \hat{M}_z^2)x\psi - x(\hat{M}_x^2 + \hat{M}_y^2 + \hat{M}_z^2)\psi) \quad (1.4)$$

$$\hat{M}_x^2 x\psi - x\hat{M}_x^2 \psi = (x\hat{M}_x^2 - x\hat{M}_x^2)\psi = 0 \quad (1.5)$$

$$\begin{aligned} (\hat{M}_y^2 x\psi - x\hat{M}_y^2 \psi) &= (x\hat{p}_z - z\hat{p}_x)^2 x\psi - x(x\hat{p}_z - z\hat{p}_x)^2 \psi = \\ &= ((x\hat{p}_z)^2 + (z\hat{p}_x)^2 - z\hat{p}_x x\hat{p}_z - x\hat{p}_z z\hat{p}_x) x\psi - \\ &\quad - x((x\hat{p}_z)^2 + (z\hat{p}_x)^2 - z\hat{p}_x x\hat{p}_z - x\hat{p}_z z\hat{p}_x) \psi = \\ &= ((z\hat{p}_x)^2 x - x(z\hat{p}_x)^2 - z\hat{p}_x x^2 \hat{p}_z + zx\hat{p}_x x\hat{p}_z - \hat{p}_z zx\hat{p}_x x + \hat{p}_z x^2 z\hat{p}_x) \psi = \\ &= ((z\hat{p}_x)^2 x - x(z\hat{p}_x)^2 + zi\hbar x\hat{p}_z + \hat{p}_z zx\hbar) \psi \end{aligned} \quad (1.6)$$

Комутатор квадрата імпульса з координатою

$$\begin{aligned} (\hat{p}_x^2 x - x\hat{p}_x^2) \psi &= -\hbar^2 \left(\frac{\partial^2 x\psi}{\partial x^2} - x \frac{\partial^2 \psi}{\partial x^2} \right) = \\ &= -\hbar^2 \left(\frac{\partial}{\partial x} (\psi + x \frac{\partial \psi}{\partial x}) - x \frac{\partial^2 \psi}{\partial x^2} \right) = \\ &= -\hbar^2 \left(2 \frac{\partial \psi}{\partial x} + x \frac{\partial^2 \psi}{\partial x^2} - x \frac{\partial^2 \psi}{\partial x^2} \right) = -2\hbar^2 \frac{\partial \psi}{\partial x} = 2i\hbar \hat{p}_x \psi \end{aligned} \quad (1.7)$$

$$[\hat{p}_x^2, \hat{x}] = 2i\hbar \hat{p}_x$$

Підставляємо комутатор (1.7) у (1.6):

$$\begin{aligned} ((z\hat{p}_x)^2 x - x(z\hat{p}_x)^2 + zi\hbar x\hat{p}_z + \hat{p}_z zx\hbar) \psi &= \\ &= (z^2 2i\hbar \hat{p}_x + i\hbar x(z\hat{p}_z + \hat{p}_z z)) \psi = \end{aligned} \quad (1.8)$$

$$\begin{aligned}
[\hat{M}_x, \hat{M}_y] \psi &= ((yp_z - zp_y)(zp_x - xp_z)\psi - (zp_x - xp_z)(yp_z - zp_y)\psi) = \\
&= (yp_z zp_x - yp_z xp_z - zp_y zp_x + zp_y xp_z - \\
&\quad - zp_x yp_z + zp_x zp_y + xp_z yp_z - xp_z zp_y) \psi = \\
&= (-i\hbar yp_x + i\hbar xp_y) \psi = i\hbar \hat{M}_z \psi
\end{aligned} \tag{1.9}$$